A quantified study of rothalpy conservation in turbomachines

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Previous studies are generally confined to establishing the fundamental validity of the assumption of rothalpy conservation and usually comment on its significance without providing any quantitative assessments to support their stated conclusion. This study attempts to quantify in terms of the order of error of temperature and velocity that may typically occur in practical turbomachine calculations involving a real fluid (one supporting both viscous stress and heat conduction) as a result of assuming that rothalpy is conserved. To this end, (constant) transport coefficients for molecular/turbulent momentum and heat conduction are included in the analysis, which derives a complete general expression for the transport of rothalpy in flow through generalised rotor blades. Flow norms are taken from a typical centrifugal compressor (being regarded as a more extreme geometry) to assess orders of error arising from each term in the rothalpy transport equation for a particle. In assessing the error significance, account is taken of how rothalpy conservation might be employed in a practical calculation scheme and to what use the error variable might subsequently be put. The general conclusion is that errors arising from the assumption of rothalpy conservation are, in practice, negligible. In particular, the effect of rotating blades, as compared to stationary ones is completely negligible, and one of the greatest errors arises from the term involving Prandtl number. If the Prandtl number is assumed to be unity, as has been the case in some classical boundary layer studies, then this very significant term vanishes from consideration.

Keywords: turbomachine; rothalpy; viscosity; heat conduction; loss; boundary layer; laminar; turbulent

Introduction

Many statements of the proposition that rothalpy is conserved in turbomachinery flows have been made over the last half century, and several authors have considered the extent of the validity of this statement. Recently, Lyman (1993) has provided a good background to past developments in this area, and this reference is commended to those wishing to pursue such developments. The validity of the proposition in the idealised model of steady relative isentropic flow was long ago established and is exact in the sense that the initial constraints of the fluid model lead algebraically to the exact mathematical result of conservation. In the present study, interest centres about its validity in steady relative flow of a real fluid through rotating passages with adiabatic walls, especially as in many instances, this validity has been assumed without proof, and where proof has been given, restrictive circumstances have applied. It greatly simplifies many flow calculation schemes in which friction is admitted if the proposition is assumed to remain true, and it is possible to devise fluid models in which it is true even where the model, although plausible from a continuum viewpoint, is clearly incorrect for a real fluid. The issue under consideration here is whether, from a practical engineering point of view, the assumption of rothalpy conservation in a real frictional fluid leads to insignificant or acceptable errors in practice. While some previous studies have addressed the issue of the untruth of the proposition in an exact sense (as defined above), there appears to be an absence of any detailed quantitative assessment of error magnitudes and their precise cause. It is this neglected aspect which forms the subject of this study where it emerges that in general the error is completely negligible, and, importantly, if the common approximation of unity Prandtl number is assumed, then one of the most significant of the error terms vanishes as a result.

It should be observed that this study is not generally amenable to numerical evaluation in the sense of making a critical study of results obtained from numerical schemes, because such schemes as are usually applied to turbomachinery analysis that include appropriate models of the physical transport processes for momentum (= molecular/turbulent viscosity) and energy (= heat conduction/convection) generally use large-eddy simulation models with grids which generate numerical error far in excess of that being here established. This paper follows a well-established

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precedent of looking analytically at a typical case in order to make error assessments. Indeed, it would have to be admitted that numerical schemes based even on isentropic flow models frequently produce localised error (e.g., in corner flow) which are gross in the sense of exhibiting a loss of several percentages of local stagnation pressure. This is particularly true in the three-dimensional (3-D) schemes in common use. The direct numerical simulation (DNS) schemes which might otherwise offer an opportunity are not commonly applied in turbomachinery applications if only because of the huge computer resources required to implement them on the scale demanded by this application. This analysis elicits levels of error which are far below the level of numerical error implicit in most numerical schemes. However, the ultimate defence of the analytical approach is that the results are uncontaminated by any numerical error whether assessable or not.

The majority of earlier studies commence by admitting real fluid effects in terms of introducing deviatoric stress and the heat flux vector into consideration without further consideration of the transport processes which give rise to these macroscopic concepts. Usually, the adiabatic fluid restriction is applied (not to be confused with the adiabatic wall condition) leaving only the deviatoric stress in consideration without regard for the real (kinetic) fluid consideration that the mechanism of molecular transport by random migration, which, in the case of momentum, leads to the concept of deviatoric stress, by the same token must admit the transport of energy leading to heat transfer by conduction. Ignoring heat transport is acceptable when considering incompressible flow if only the mechanical flow features (to the exclusion of thermodynamic ones) are of interest, because (assuming temperature-independent transport coefficients) temperature then does not modify the mechanical flow field, but analyses for compressible flow often also follow this development where the thermodynamic interaction may not be negligible. Admission of momentum transport (deviatoric stress) while denying energy transport (heat conduction) represents an inconsistent model of a real fluid. Such an example of a real fluid inconsistency exists in the proposal of the rothalpy-conserving loss model of Bosman and Marsh (1974), which is consistent at the continuum level and is achieved not by the conduction of the dissipated energy as interpreted by Lyman (1993) (since the model was explicitly adiabatic in omitting heat flux) but by the consideration that the energy lost by a particle due to work done in overcoming the drag force of the surrounding flow is replaced by the dissipation energy generated by the continuous distortion of the particle. The present study introduces transport coefficients for momentum and energy transfer and proceeds to consider on a term-by-term basis the typical physical magnitudes of error which would result from the assumption of rothalpy conservation in respect to using this equation to determine either the temperature or speed of the flow at a point in a typical centrifugal compressor. In particular, two-dimensional (2-D) calculation schemes based on streamline curvature techniques have commonly used (explicitly or implicitly) rothalpy conservation as a basis for determining flow speed, but it has also been used in some 3-D calculation schemes, whereas, in 2-D streamfunction

Notation*

C_p	specific heat at constant pressure
C_v	specific heat at constant volume
h	static enthalpy
i, j, k	general Cartesian coordinate directions
$\mathbf{F}_\mu$	resultant force vector on particle due to viscous stresses
I	rothalpy
k	thermal conductivity
l	typical length scale
M_r	relative Mach number
O	principal centre of curvature of streamline
p	static pressure
P	point under consideration in flow field
Pe	Peclet number
Pr	Prandtl number
$\mathbf{q}$	heat flux density vector = directional heat transfer per unit area per unit time
Q_{in}	heat transfer to particle
$\mathbf{R}_{cs}$	vector principal radius of curvature of streamline
Re	Reynolds number
Ro	Rosby number
s	entropy
t	time
T	static temperature
$\mathbf{U}$	velocity vector of fixed point P in the rotating frame
$\mathbf{V}$	fluid velocity vector relative to stationary frame
$\mathbf{W}$	fluid velocity vector relative to rotating frame
x	Cartesian coordinate

*All extensive properties are on a unit mass basis. In the text "rotating frame" refers to a reference frame rotating with the blades. All algebraic equations assume consistent units and, therefore, contain no units conversion constants. All dimensional numerical values are in MKS units.

Greek

γ	isentropic index (= specific heat ratio C_p/C_v)
κ	thermal diffusivity
μ	dynamic viscosity
ν	kinematic viscosity
ξ	power dissipation by deviatoric stresses on particle
ρ	fluid density
σ	deviatoric (= viscous/turbulent) stress
ϕ	angle
$\boldsymbol{\omega}$	angular velocity of rotating frame

Subscripts

1,2,3	specified Cartesian components
i, j, k	general Cartesian components
t	indicating time constant derivative

Superscripts

$\tilde{}$	(tilde) norm value
$'$	(prime) normalised value

Operators

$\bullet$	scalar product
$\times$	vector product
Δ	increment in value on passage through the blade row
∇	vector gradient operator
$\partial/\partial t$	temporal derivative for fixed observer in rotating frame
D/Dt	temporal derivative for particle observer
δ_{ij}	unit tensor
$\| \|$	modular value
$\cong$	is of the decimal order of
$\approx$	has the approximate value

and 3-D schemes, it usually forms a basis for temperature determination. The attraction of its use lies in the saving of calculations consequent on not having to solve for the variable by numerical integration of an additional discretised differential or integral equation.

Analysis

The introductory forms of the equations below, which are given in terms of the relative velocity $\mathbf{W}$, are well established and are presented without derivation. They conform to standard turbomachinery calculation practice wherein the flow is considered steady relative to the blade row, fluctuations due to other local blade rows in relative motion being neglected. Transport coefficients are assumed constant throughout the paper.

Continuity

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \mathbf{W} = 0 \tag{1}$$

Equation of motion

For blades rotating at constant angular velocity $\boldsymbol{\omega}$

$$\frac{D\mathbf{W}}{Dt} - \nabla\left(\frac{U^2}{2}\right) + 2\boldsymbol{\omega} \times \mathbf{W} + \frac{1}{\rho}\nabla p = \mathbf{F}_\mu \tag{2}$$

A form of the relative streamwise component of this equation is obtained by operating on it with the scalar product of the relative velocity $\mathbf{W}$, which for steady relative flow, i.e.

$$\left(\frac{\partial}{\partial t} = 0\right)$$

and observing A10, is

$$\frac{D}{Dt}\left(\frac{W^2 - U^2}{2}\right) + \frac{1}{\rho}\frac{Dp}{Dt} = \mathbf{W} \cdot \mathbf{F}_\mu \tag{3}$$

Definition of rothalpy (I)

$$h = C_p T$$

$$I \equiv h + \frac{W^2 - U^2}{2} \tag{4}$$

1st law

$$\frac{Dh}{Dt} = \frac{DQ_{\text{in}}}{Dt} + \frac{1}{\rho}\frac{Dp}{Dt} + \xi \tag{5}$$

hence by Equation 4

$$\frac{DI}{Dt} \equiv \frac{D}{Dt}\left(h + \frac{W^2 - U^2}{2}\right)$$

$$= \frac{DQ_{\text{in}}}{Dt} + \frac{D}{Dt}\left(\frac{W^2 - U^2}{2}\right) + \frac{1}{\rho}\frac{Dp}{Dt} + \xi \tag{6}$$

Fourier heat conduction

$$\mathbf{q} = -k\nabla T \tag{7}$$

which for constant k yields

$$\frac{DQ_{\text{in}}}{Dt} = -\frac{1}{\rho}\nabla \cdot \mathbf{q} = \frac{k}{\rho}\nabla^2 T \tag{8}$$

observing Equation 4 for constant C_p

$$\frac{DQ_{\text{in}}}{Dt} = \kappa \nabla^2 h \tag{9}$$

$$\kappa \equiv \frac{k}{\rho C_p}$$

Equation for the transport of rothalpy

Substituting 3 and 9 in Equation 6 yields

$$\frac{DI}{Dt} = \kappa \nabla^2\left(I + \frac{U^2}{2} - \frac{W^2}{2}\right) + \mathbf{W} \cdot \mathbf{F}_\mu + \xi \tag{10}$$

and introducing the Prandtl number $\Pr \equiv (\nu/k)$ while observing the details of Appendix A, we obtain the following form:

$$\frac{DI}{Dt} = \kappa(\nabla^2 I + 2\omega^2) + \nu\left(1 - \frac{1}{\Pr}\right)\nabla^2\left(\frac{W^2}{2}\right)$$
$$+ \nu\left[\frac{1}{3}\frac{D}{Dt}\left(-\frac{1}{\rho}\frac{D\rho}{Dt}\right) + \frac{\partial W_i}{\partial x_j}\frac{\partial W_j}{\partial x_i} - \frac{2}{3}\left(\frac{1}{\rho}\frac{D\rho}{Dt}\right)^2\right] \tag{11}$$

where

$$\frac{\partial W_i}{\partial x_j}\frac{\partial W_j}{\partial x_i} \equiv \left(\frac{\partial W_1}{\partial x_1}\right)^2 + \left(\frac{\partial W_2}{\partial x_2}\right)^2 + \left(\frac{\partial W_3}{\partial x_3}\right)^2$$
$$+ 2\left(\frac{\partial W_1}{\partial x_2}\frac{\partial W_2}{\partial x_1} + \frac{\partial W_2}{\partial x_3}\frac{\partial W_3}{\partial x_2} + \frac{\partial W_3}{\partial x_1}\frac{\partial W_1}{\partial x_3}\right) \tag{12}$$

and a physical interpretation of this term may be seen in Appendices A and B.

This equation can be given the more general form below by nondimensionalising in terms of a relative velocity norm $\tilde{W}$ and length norm $\tilde{l}$, which together provide a time norm

$$\tilde{t} \equiv \frac{\tilde{l}}{\tilde{W}} \tag{13}$$

Because I is an energy quantity with an arbitrary datum because of the temperature; i.e., practical calculations would not be disturbed by assuming I to be zero at flow entry, $\tilde{W}^2$ is a preferred divisor for nondimensionalising Equation 11. Thus defining a normalised time

$$t' \equiv \frac{t}{\tilde{t}} \tag{14}$$

then

$$\frac{D}{Dt'} = \frac{1}{\tilde{t}}\frac{D}{Dt} \tag{15}$$

which with the further definitions of primed variables indicating the normalised values

$$W' \equiv \frac{W}{\tilde{W}}$$

$$x' \equiv \frac{x}{\tilde{l}}$$

$$\nabla'^2 \equiv \tilde{l}^2 \nabla^2$$

$$I' \equiv \frac{I}{\tilde{W}^2} \tag{16}$$

and the predefined Reynolds Re, Prandtl Pr, Peclet Pe, and Rosby Ro numbers (assuming constant transport properties)

$$\tilde{\mathrm{Re}} \equiv \frac{\tilde{\rho}\tilde{W}\tilde{l}}{\mu}$$

$$\tilde{\mathrm{Pr}} = \mathrm{Pr} \equiv \frac{\mu C_p}{k}$$

$$\tilde{\mathrm{Pe}} \equiv \tilde{\mathrm{Pr}}\,\tilde{\mathrm{Re}}$$

$$\tilde{\mathrm{Ro}} \equiv \frac{\omega \tilde{l}}{\tilde{W}} \tag{17}$$

enables Equation 11 to be written

$$\frac{DI'}{Dt'} = \frac{1}{\mathrm{Pe}}\left\{(\nabla'^2 I' + 2\,\mathrm{Ro}^2) + (\mathrm{Pr}-1)\nabla'^2\left(\frac{W'^2}{2}\right) + \mathrm{Pr}\left[\frac{1}{3}\frac{D}{Dt'}\left(-\frac{1}{\rho}\frac{D\rho}{Dt'}\right) + \frac{\partial W_i'}{\partial x_j'}\frac{\partial W_j'}{\partial x_i'} - \frac{2}{3}\left(\frac{1}{\rho}\frac{D\rho}{Dt'}\right)^2\right]\right\} \tag{18}$$

Equation 11 when applied in the stationary reference frame (where $I = h_0$ and $\mathbf{W} = \mathbf{V}$) with the usual thin collateral boundary-layer approximation imposed will be recognised as the familiar result

$$\frac{Dh_0}{Dt} = \kappa\left(\frac{\partial^2 h_0}{\partial y^2}\right) + \nu\left(1 - \frac{1}{\mathrm{Pr}}\right)\frac{\partial^2}{\partial y^2}\left(\frac{V^2}{2}\right) \tag{19}$$

wherein for $\mathrm{Pr} = 1$, y being displacement normal to the wall, the well-known observation is recovered that h_0 is uniform is a solution, and therefore h_0 is conserved (Shapiro (1954); Goldstein (1938).

It should be observed that entropy is absent from the mathematical development here presented, because the concept is not relevant to the discussion as entropy is a secondary level concept resulting from the primary level concepts of dissipation rate and heat transfer which reflect fundamental physical transport processes; namely,

$$T\frac{Ds}{Dt} = \xi + \frac{DQ_{\mathrm{in}}}{Dt} \tag{20}$$

Observations

It can be seen that in the isentropic case where $(\nu, \kappa) = 0$ then Equation 11 becomes $(DI/Dt) = 0$ showing that a particle then experiences no change in the rothalpy I. It is then tempting to present the effect of the transport processes ν and κ directly in terms of the change that does occur in I when these processes are admitted, but this is not very meaningful, because this value provides no indication of the significance of the error on practical engineering calculations.

The analysis here seeks the order-of-magnitude error by which each term in Equation 11 would modify the estimate of the increment Δ of the flow through the blade row, of temperature or velocity, as a result of assuming rothalpy conservation. In general, the value of I is used to determine the temperature T, but it is used alternatively in 2-D schemes employing the streamline curvature technique to determine the relative speed W, the direction being determined by streamline fitting to satisfy the continuity equation. The presentation of the amount by which I itself may change in a typical flow is not of value, because in practical calculations, it is not the subject of calculation and serves only as an agent as described above. Moreover, its percentage change is valueless, because being an energy, it is a quantity with an arbitrary datum, and in the incompressible case, the temperature term becomes a pressure term, and in hydraulic and low-speed fan applications, both absolute and gauge pressure are in common use, so here the alternative datum would affect the percentage change in I by a large factor. By Equations 18 and 4

$$\frac{DI'}{Dt'} \equiv \frac{1}{\tilde{W}^2}\frac{D}{Dt'}\left(C_pT + \frac{W^2 - U^2}{2}\right)$$

$$\approx \frac{1}{\tilde{W}^2}\frac{\Delta}{\Delta t'}\left(C_pT + \frac{W^2 - U^2}{2}\right) = (\text{RHS terms}) \tag{21}$$

giving rise to a speed error

$$\frac{1}{\tilde{W}^2}\Delta\left(\frac{W^2}{2}\right) \approx \frac{\Delta W}{\tilde{W}} \approx \Delta t'(\text{RHS terms}) \tag{22}$$

or a temperature error

$$\frac{\Delta(T)}{\tilde{T}} \approx \frac{\tilde{W}^2}{\tilde{T}}\frac{\Delta t'}{C_p}(\text{RHS terms}) \tag{23}$$

hence, from Equations 22 and 23

$$\frac{\Delta T}{\tilde{T}} \approx \frac{\tilde{W}^2}{C_p\tilde{T}}\frac{\Delta W}{\tilde{W}} = (\gamma - 1)\tilde{M}_r^2\frac{\Delta W}{\tilde{W}} \tag{24}$$

where the relative Mach number norm is defined by ($\tilde{T}$ necessarily being the absolute value)

$$\tilde{M}_r \equiv \frac{\tilde{W}}{\sqrt{\gamma R\tilde{T}}} \tag{25}$$

The subsequent results are based on values that could correspond to a centrifugal air compressor achieving a density ratio of 2.0, at 10,000 rpm and having a meridional path length of the order of 10 cm and mean relative speed of the order of 100 m/s at 300 K (Table 1).

Table 1 Norm values used in subsequent observations

μ	$\cong 10^{-6}$
$\tilde{\rho}$	$\cong 1$
Δt	$\cong 10^{-3}$
$\tilde{l}, \tilde{R}_{cs}$	$\cong 10^{-1}$
ω	$\cong 10^{3}$
C_p	$\cong 10^{3}$
κ	$\cong 10^{-6}$
$\nabla, \frac{1}{\tilde{l}}$	$\cong 10$
$\Delta\rho$	$\cong 1$
$\tilde{W}$	$\cong 10^{2}$
(Pr − 1)	$\cong 1$

$\cong$ means (of the decimal order of); $\Delta \equiv$ (incremental value in passage through blade row).

The current norms lead by means of Equations 17, 24 and 25 to the values in Table 2. For the norms as defined at Equations 13, 14, and 16 $\Delta t' \cong 1$ and that $\Delta x' \cong 1$ (always) and will always lead by Table 1 to the algebraic order of magnitude approximations given in the left-hand column of Table 3. The right-hand column of the table states the values of the order of magnitude for the current set of norm values and assumed increments.

The term $(\nabla'^2 I')$ requires special attention because consideration of the remaining terms shows, Equation 18, that $\Delta I' \cong \Delta t' \cdot$ (largest right-hand term) so that the contribution to $(\nabla'^2 I')$ from the flow direction is from Table 2, of the order of $\Delta I' \cong 1/\tilde{\text{P}}\text{e} = 10^{-7}$, a second-order error as anticipated from the fact that $\Delta I \equiv 0$ in the isentropic flow. If the upstream flow is uniform then this is the resulting error level as the contribution to $(\nabla'^2 I')$ from the cross-flow direction would then be zero. Any significant contribution will then only arise from an upstream cross-flow variation. For this consideration, a variation of $\delta T = 10$°C is assumed corresponding to $\delta I' = C_p \delta T/\tilde{W}^2 \cong 10^3 10/(10^2)^2 = 1$ and leading to $\nabla'^2 I' \cong [\delta I'/(\Delta x')^2] = 1$.

For the core flow errors resulting from each right-hand term in Equation 18 are now summarised in Table 4 below where the temperature errors are determined from the velocity errors by Equation 24, while observing Table 2. Note that from Equation 24, the temperature must now be the absolute value.

For the core flow the errors are seen to be completely negligible and

$$\frac{DI}{Dt} = 0 \tag{26}$$

i.e., rothalpy is conserved to all practical engineering intents and purposes. If the flow emanates from a uniform upstream source

Table 2 Values for the standard flow control parameters

$\tilde{\text{R}}\text{e}$	$\cong 10^{7}$
$\tilde{\text{P}}\text{r}$	$\cong 1$
$\tilde{\text{P}}\text{e}$	$\cong 10^{7}$
$\tilde{\text{R}}\text{o}$	$\cong 1$
$(\gamma - 1)\tilde{M}_r^2$	$\cong 10^{-1}$

Table 3 Expressions and values for norms appearing in Equation 18

$\frac{1}{\rho}\frac{D\rho}{Dt'} \approx \frac{1}{\tilde{\rho}}\frac{\Delta\rho}{\Delta t'} \cong \frac{\Delta\rho}{\tilde{\rho}}$	1
$\frac{D}{Dt'}\left(\frac{1}{\rho}\frac{D\rho}{Dt'}\right) \approx \frac{1}{\tilde{\rho}}\frac{\Delta(\Delta\rho)}{(\Delta t')^2} \cong \frac{\Delta\rho}{\tilde{\rho}}$	1
$\frac{\partial W_i'}{\partial x_j'} \approx \frac{\Delta W'}{\Delta x'} \cong \Delta W'$	1
$\nabla'^2 W'^2 \approx \frac{\partial^2 W'^2}{\partial x'^2} \approx \frac{\Delta(\Delta W'^2)}{(\Delta x')^2} \cong \Delta W'^2$	1

where $\nabla^2 I = 0$, then the entire flow field is uniform in I and if, as is typically assumed for the upstream condition of an $S1$ stream surface, the flow is axisymmetric, then that stream surface has a uniform value of I. These observations remain true even when the transport coefficients, and hence Pe, are scaled up by a factor of 100 to account for strong free-stream turbulence.

In the boundary layers, the relevant length scale (i.e., boundary-layer thickness) $\tilde{l}$ is typically 100 times smaller than the particle (not molecular) mean path length scale, so the situation here is somewhat different, and the effect of this is indicated in Table 5 for laminar boundary layers, however in the context of boundary-layer calculation schemes, the error in speed is not quoted, because in such a context, I would never be used to determine the speed.

In Table 5, the order of the term in $[(\partial W_i'/\partial x_j')(\partial W_j'/\partial x_i')]$ is based on the more physically interpretable quantity $-2(\mathbf{R}_{cs}/R_{cs}^2) \cdot \nabla(W^2/2)$, see Equation B4. The effects are still negligible, and the temperature may be determined from the assumption of rothalpy conservation down to the wall with negligible error to all engineering intents and purposes; i.e., 10^{-4}, based on the above norms.

In the case of turbulent boundary layers, taking a Boussinesq view, the transport coefficients ν, κ and hence Pe, may need to be enhanced by a factor of 100, and, of course, there are now additional terms (not treated here) involving their derivatives. The assumption of constancy is allowable, because few if any calculation schemes take account of variable transport coefficients, which, themselves, vary according to empirical origin.

Table 4 Order of effects of individual terms on temperature or velocity for the core flow

Right-hand term	$\frac{\lvert\Delta T\rvert}{\tilde{T}}$	$\frac{\lvert\Delta W\rvert}{\lvert\tilde{W}\rvert}$
$\tilde{\text{P}}\text{e}^{-1} * 2\tilde{\text{R}}\text{o}^2$	10^{-8}	10^{-7}
$\tilde{\text{P}}\text{e}^{-1} * (\tilde{\text{P}}\text{r} - 1)\nabla'^2\left(\frac{W'^2}{2}\right)$	10^{-8}	10^{-7}
$\tilde{\text{P}}\text{e}^{-1} * \frac{\tilde{\text{P}}\text{r}}{3}\frac{D}{Dt'}\left(-\frac{1}{\rho}\frac{D\rho}{Dt'}\right)$	10^{-8}	10^{-7}
$\tilde{\text{P}}\text{e}^{-1} * \tilde{\text{P}}\text{r}\frac{\partial W_i'}{\partial x_j'}\frac{\partial W_j'}{\partial x_i'}$	10^{-8}	10^{-7}
$\tilde{\text{P}}\text{e}^{-1} * \tilde{\text{P}}\text{r}\left(-\frac{2}{3}\left(\frac{1}{\rho}\frac{D\rho}{Dt'}\right)^2\right)$	10^{-8}	10^{-7}
$\tilde{\text{P}}\text{e}^{-1} * \nabla'^2 I'$	10^{-8}	10^{-7}

Table 5 Order of effects of individual terms on temperature for the laminar boundary layer

Right-hand term	$\frac{\lvert\Delta T\rvert}{\tilde{T}}$
$\tilde{P}e^{-1}*2\tilde{R}o^2$	10^{-8}
$\tilde{P}e^{-1}*(\tilde{P}r-1)\nabla'^2\left(\frac{W'^2}{2}\right)$	10^{-4}
$\tilde{P}e^{-1}*\frac{\tilde{P}r}{3}\frac{D}{Dt'}\left(-\frac{1}{\rho}\frac{D\rho}{Dt'}\right)$	10^{-8}
$\tilde{P}e^{-1}*\tilde{P}r\frac{\partial W_i'}{\partial x_j'}\frac{\partial W_j'}{\partial x_i'}$	10^{-4}
$\tilde{P}e^{-1}*\tilde{P}r\left(-\frac{2}{3}\left(\frac{1}{\rho}\frac{D\rho}{Dt'}\right)^2\right)$	10^{-8}
$\tilde{P}e^{-1}*\nabla'^2 I'$	10^{-4}

Table 6 is modified as below.

Here alone the dominant second, fourth, and sixth terms in Table 6 may become locally significant, yielding corrections of the order of 1°C, but again, the blade row rotation term in Ro remains insignificant. It should be noted here that the common approximation Pr = 1.0 would have eliminated one of the dominant terms from consideration. Even this largest error is of little significance, because its influence on density would be of the order of $(\Delta T/\tilde{T}) = 10^{-2}$, and although 1°C would be significant on the mean temperature rise, the small boundary-layer mass flux would reduce it by a factor of order [(boundary layer)/(core) cross-sectional areas] $= (\tilde{l}\cdot\tilde{l}/100)(/(\tilde{l}^2) = 10^{-2}$. Again, then, the general conclusion is that with negligible error to all practical engineering intents and purposes rothalpy may be considered to be conserved.

Conclusions

In general, the order of magnitude of error to the estimated speed or temperature at a point in the flow field as a result of assuming conservation of rothalpy is negligible to all engineering intents and purposes. This observation is likely to be true for both the core flow and the boundary-layer flow when account is taken of what property is likely to be determined using the assumption and to what use that property will be put. Clearly, the assumption could not be used in a boundary-layer context to determine near-wall speeds of relative flow, but rothalpy is and never would be used for this purpose in any practical calculation scheme. The effect of blade rotation is at all times negligible, indicating that the observations are common to rotor and stator. Bearing in mind that the above norms imply a mass flow rate in the core of the order of 10^2 times the flow through the boundary layers, it is apparent that the effect of errors in boundary-layer values, on the total integrated mass flow values of speed and temperature, as determined from rothalpy conservation will be negligible.

These findings are analogous with conclusions drawn from the well-established condition of uniform stagnation temperature in boundary layers over adiabatic walls with Prandtl number unity in the stationary reference frame. For uniform stagnation temperature in the stationary frame, we can substitute the more general uniform "rothalpy" in the rotating frame.

In more extreme cases, as in hot turbine flows, larger cross-stream temperature variations may lead to temperature errors which become unacceptable. However, following the methodology here presented, sensible estimates can be made a priori of the likely scale of the predicted temperature error.

Table 6 Order of effects of individual terms on temperature for the turbulent boundary layer

Right-hand term	$\frac{\lvert\Delta T\rvert}{\tilde{T}}$
$\tilde{P}e^{-1}*2\tilde{R}o^2$	10^{-6}
$\tilde{P}e^{-1}*(\tilde{P}r-1)\nabla'^2\left(\frac{W'^2}{2}\right)$	10^{-2}
$\tilde{P}e^{-1}*\frac{\tilde{P}r}{3}\frac{D}{Dt'}\left(-\frac{1}{\rho}\frac{D\rho}{Dt'}\right)$	10^{-6}
$\tilde{P}e^{-1}*\tilde{P}r\frac{\partial W_i'}{\partial x_j'}\frac{\partial W_j'}{\partial x_i'}$	10^{-2}
$\tilde{P}e^{-1}*\tilde{P}r\left(-\frac{2}{3}\left(\frac{1}{\rho}\frac{D\rho}{Dt'}\right)^2\right)$	10^{-6}
$\tilde{P}e^{-1}*\nabla'^2 I'$	10^{-2}

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Appendix A

$$\mathbf{F}_\mu \equiv \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j} \equiv \quad \text{(resultant force on particle due to deviatoric stresses, per unit mass)} \tag{A1}$$

$$\xi \equiv \frac{\sigma_{ij}}{\rho}\frac{\partial V_i}{\partial x_j} \equiv \quad \text{(power dissipated on particle by deviatoric stresses, per unit mass)} \tag{A2}$$

Note that

$$\nabla^2\left(\frac{U^2}{2}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial}{\partial r}\right)\frac{\omega^2 r^2}{2}\right] = 2\omega^2 \tag{A3}$$

The viscous power dissipation due to the distortionless motion of the rotating frame is zero; i.e., $\sigma_{ij}\,\partial U_i/\partial x_j = 0$, hence in Equation A2

$$\xi \equiv \frac{\sigma_{ij}}{\rho}\frac{\partial V_i}{\partial x_j} = \frac{\sigma_{ij}}{\rho}\left(\frac{\partial W_i}{\partial x_j} + \frac{\partial U_i}{\partial x_j}\right) = \frac{\sigma_{ij}}{\rho}\frac{\partial W_i}{\partial x_j} \tag{A4}$$

and

$$\mathbf{W}\cdot\mathbf{F}_\mu + \xi = \frac{W_i}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j} + \frac{\sigma_{ij}}{\rho}\frac{\partial W_i}{\partial x_j} \equiv \frac{1}{\rho}\frac{\partial(W_i\sigma_{ij})}{\partial x_j} \tag{A5}$$

Considering a Newtonian fluid

$$\sigma_{ij} = \mu\left[\left(\frac{\partial W_i}{\partial x_j} + \frac{\partial W_j}{\partial x_i}\right) - \frac{2}{3}\left(\frac{\partial W_k}{\partial x_k}\right)\delta_{ij}\right] \tag{A6}$$

then with constant μ and observing that $W_i\delta_{ij} = W_j$

$$\frac{\partial(W_i\sigma_{ij})}{\partial x_j} = \mu\left[\frac{\partial}{\partial x_j}\left(W_i\frac{\partial W_i}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(W_i\frac{\partial W_j}{\partial x_i}\right) - \frac{2}{3}\frac{\partial}{\partial x_j}\left(W_j\frac{\partial W_k}{\partial x_k}\right)\right]$$

$$= \mu\left\{\frac{\partial^2}{\partial x_j^2}\left(\frac{W_i^2}{2}\right) + W_i\frac{\partial}{\partial x_i}\left(\frac{\partial W_j}{\partial x_j}\right) + \frac{\partial W_i}{\partial x_j}\frac{\partial W_j}{\partial x_i}\right.$$

$$\left. - \frac{2}{3}\left[W_j\frac{\partial}{\partial x_j}\left(\frac{\partial W_k}{\partial x_i}\right) + \frac{\partial W_j}{\partial x_j}\frac{\partial W_k}{\partial x_k}\right]\right\}$$

$$= \mu\left[\frac{\partial^2}{\partial x_j^2}\left(\frac{W_i^2}{2}\right) + \frac{1}{3}W_i\frac{\partial}{\partial x_i}\left(\frac{\partial W_j}{\partial x_j}\right) + \frac{\partial W_i}{\partial x_j}\frac{\partial W_j}{\partial x_i}\right.$$

$$\left. - \frac{2}{3}\frac{\partial W_j}{\partial x_j}\frac{\partial W_k}{\partial x_k}\right] \tag{A7}$$

however, because

$$\frac{\partial^2}{\partial x_j^2}\left(\frac{W_i^2}{2}\right) \equiv \nabla^2\left(\frac{W^2}{2}\right) \tag{A8}$$

and by Equation 1

$$\frac{\partial W_j}{\partial x_j} \equiv \frac{\partial W_k}{\partial x_k} \equiv \nabla \cdot \mathbf{W} = -\frac{1}{\rho}\frac{D\rho}{Dt} \tag{A9}$$

and for steady relative flow

$$W_i\frac{\partial}{\partial x_i} \equiv W_j\frac{\partial}{\partial x_j} \equiv \mathbf{W} \cdot \nabla = \frac{D}{Dt} \tag{A10}$$

then

$$\frac{\partial(W_i\sigma_{ij})}{\partial x_j} = \mu\left[\nabla^2\left(\frac{W^2}{2}\right) + \frac{1}{3}\frac{D}{Dt}\left(-\frac{1}{\rho}\frac{D\rho}{Dt}\right)\right.$$

$$\left. + \frac{\partial W_i}{\partial x_j}\frac{\partial W_j}{\partial x_i} - \frac{2}{3}\left(\frac{1}{\rho}\frac{D\rho}{Dt}\right)^2\right] \tag{A11}$$

The term

$$\frac{\partial W_i}{\partial x_j}\frac{\partial W_j}{\partial x_i} \equiv \left(\frac{\partial W_i}{\partial x_j}\right)^2_{i=j} + \left(\frac{\partial W_i}{\partial x_j}\frac{\partial W_j}{\partial x_i}\right)_{i\neq j} \tag{A12}$$

where

$$\left(\frac{\partial W_i}{\partial x_j}\right)^2_{i=j} = \left(\frac{\partial W_i}{\partial x_1}\right)^2 + \left(\frac{\partial W_2}{\partial x_2}\right)^2 + \left(\frac{\partial W_3}{\partial x_3}\right)^2 \tag{A13}$$

is the (sum of the squares of the normal strain rates) and

$$\left(\frac{\partial W_i}{\partial x_j}\frac{\partial W_j}{\partial x_i}\right)_{i\neq j} \equiv 2\left(\frac{\partial W_1}{\partial x_2}\frac{\partial W_2}{\partial x_1} + \frac{\partial W_2}{\partial x_3}\frac{\partial W_3}{\partial x_2} + \frac{\partial W_3}{\partial x_1}\frac{\partial W_1}{\partial x_3}\right) \tag{A14}$$

As shown in appendix B, one of the terms in the brackets on the right hand side can be interpreted as

$$\left(\frac{\partial W_i}{\partial x_j}\frac{\partial W_j}{\partial x_i}\right)_{i\neq j} = -\frac{\mathbf{R}_{cs}}{R_{cs}^2}\cdot\nabla\left(\frac{W^2}{2}\right) \tag{A15}$$

i.e., 2 (minus the gradient of the relative kinetic energy in the direction of the principal radius of curvature of the streamline, divided by that radius of curvature) where $R_{cs} \equiv$ principal radius of curvature of the streamline. In a 2-D, plane flow, as in unskewed boundary-layer flow, the other two terms would then be zero.

Appendix B

Let 1 = the local (= at point P) flow direction (W) and 2 = the direction indicated by the current (= at time t and point P) principal radius of curvature vector ($\mathbf{R}_{cs}$) of the streamline. This pair (1,2) define orthogonal coordinates (i.e., $\mathbf{W}\cdot\mathbf{R}_{cs} = 0$). Now take an origin of coordinates (O) at the streamline principal centre of curvature, as shown in Figure B1. Then

$$\left(\frac{\partial W_1}{\partial x_2}\right)_t = (\text{gradient of } W \text{ in direction of } \mathbf{R}_{cs}) = \frac{\mathbf{R}_{cs}}{|R_{cs}|}\cdot\nabla W \tag{B1}$$

$$\left(\frac{\partial W_2}{\partial x_1}\right)_t = \frac{-|\mathbf{W}|\,d\phi}{|\mathbf{R}_{cs}|\,d\phi} = -\frac{|\mathbf{W}|}{|\mathbf{R}_{cs}|} \tag{B2}$$

hence

$$\left(\frac{\partial W_1}{\partial x_2}\frac{\partial W_2}{\partial x_1}\right) = -\frac{|\mathbf{W}|}{|\mathbf{R}_{cs}|}\frac{\mathbf{R}_{cs}}{|\mathbf{R}_{cs}|}\cdot\nabla|\mathbf{W}| = -\frac{\mathbf{R}_{cs}}{R_{cs}^2}\cdot\nabla\left(\frac{W^2}{2}\right) \tag{B3}$$

For a thin, collateral boundary layer on a convex surface $\mathbf{R}_{cs}\cdot\nabla(W^2/2) > 0$, because W increases in the direction of $\mathbf{R}_{cs}$ and so $(\partial W_1/\partial x_2)(\partial W_2/\partial x_1) < 0$. The reverse is true for a concave surface.

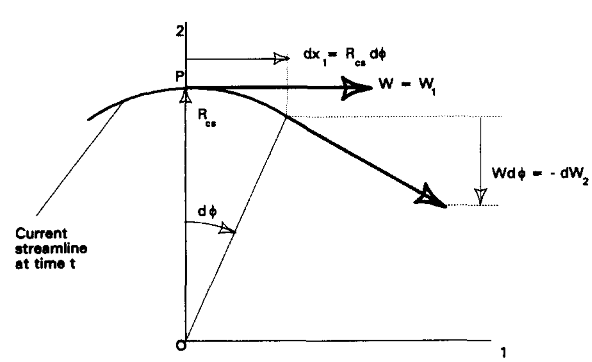

Figure B1 Illustrating the local streamline at point P and time t, in relation to the selected coordinates (1,2) and appropriate velocity vectors

For a 2-D plane flow $W_3 = 0$, $\partial/\partial x_3 = 0$ *so*

$$\left(\frac{\partial W_i}{\partial x_j}\frac{\partial W_j}{\partial x_i}\right)_{i \neq j} \equiv 2\left(\frac{\partial W_1}{\partial x_2}\frac{\partial W_2}{\partial x_1} + \frac{\partial W_2}{\partial x_3}\frac{\partial W_3}{\partial x_2} + \frac{\partial W_3}{\partial x_1}\frac{\partial W_1}{\partial x_3}\right)$$

$$= 2\frac{\partial W_1}{\partial x_2}\frac{\partial W_2}{\partial x_1} = -2\frac{\mathbf{R}_{cs}}{R_{cs}^2} \cdot \nabla\left(\frac{W^2}{2}\right) \qquad \text{(B4)}$$

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